Séminaire Lotharingien de Combinatoire **78B** (2017)
Article #75, 12 pp.

Proceedings of the 29th Conference on Formal Power Series and Algebraic Combinatorics (London)

From dimers to tensor invariants

Chris Fraser[1], Thomas Lam[2] and Ian Le[3]

[1]*Indiana University–Purdue University, Indianapolis*
[2]*University of Michigan, Ann Arbor*
[3]*Perimeter Institute, Waterloo, Ontario*

Abstract. We formulate a higher-rank version of the *boundary measurement map* for weighted planar bipartite *networks* in the disk. It sends a network to a linear combination of SL_r *webs*, and is built upon the r-fold dimer model on the network. When r is 1, our map is a reformulation of Postnikov's boundary measurement used to coordinatize positroid strata. When r is 2 or 3, it is a reformulation of the SL_2 and SL_3 *web immanants* defined by the second author. The basic result is that the higher rank map factors through Postnikov's map. As an application, we deduce generators and relations for the space of SL_r webs, reproving a result of Cautis-Kamnitzer-Morrison. We establish compatibility between our map and restriction to positroid strata, and thus between webs and total positivity.

Résumé. Nous formulons une version de rang supérieur de la carte de mesure des limites pour les réseaux biparités planaires pondérés dans le bureau. Il envoie un réseau á une combinaison linéaire de bandes pour SL_r et est construit sur le modéle dimére á pli r sur le réseau. Lorsque r est 1, notre carte est une reformulation de la mesure de fronti'ere de Postnikov utilisée pour coordonner les strates positroÃ¯des. Lorsque r est 2 ou 3, il s'agit de la reformulation des immanents de bande SL_2 et SL_3 définis par le second auteur. Le résultat de base est que les facteurs de carte de rang supérieur par la carte de Postnikov. En tant qu'application, on déduit des générateurs et des relations pour l'espace des bandes SL_r, reprochant un résultat de Cautis-Kamnitzer-Morrison. Nous établissons la compatibilité entre notre carte et la restriction aux strates positroÃ¯des, et donc entre les bandes et la positivité totale.

Keywords: dimer, web, boundary measurement, positroid, Grassmannian

Introduction

The Grassmannian $\mathrm{Gr}(k,n)$ of k-planes in $\mathbb{C}^n$ is an algebraic variety which has been well-loved in algebraic combinatorics. This paper links two combinatorial tools – briefly, *dimer configurations* and *webs* – which have been used to study $\mathrm{Gr}(k,n)$ and its (homogeneous) coordinate ring $\mathbb{C}[\mathrm{Gr}(k,n)]$. Both approaches have a similar flavor – each involves certain planar diagrams in a disk, and each relies heavily on local diagrammatic moves/relations amongst such diagrams. We show that this resemblance is not coincidental, and that these approaches are in some sense dual to each other. Moreover, statements on each side can be translated to the other to give meaningful consequences.

The first approach starts with a choice of *network* N, meaning a planar bipartite graph in the disk whose edges are weighted by nonzero complex numbers. Such a network comes with two parameters: n (the number of boundary vertices) and k (the *excedance*). The key operation is Postnikov's *boundary measurement map*

$$N \mapsto (\Delta_I(N))_{I \in \binom{[n]}{k}} \in \mathrm{Gr}(k,n) \tag{0.1}$$

sending a network N to its $\binom{n}{k}$ *boundary measurements*. Each boundary measurement $\Delta_I(N)$ is a complex number obtained by summing over *dimer configurations of* N whose *boundary data* is I. The striking feature is that for any network, the $\Delta_I(N)$ satisfy the well-known *Plücker relations*, so the image of the boundary measurement is a point in the Grassmannian.

The second approach considers a distinguished class of functions on $\mathrm{Gr}(r,n)$, indexed by planar diagrams known as SL_r-*webs*. Let us consider the space $\mathcal{W}(r,n) = \mathrm{Hom}_{\mathrm{SL}_r}((\mathbb{C}^r)^{\otimes n}, \mathbb{C})$. The homogeneous coordinate ring $\mathbb{C}[\mathrm{Gr}(r,n)]$ can be viewed as an algebra of SL_r-invariants, and $\mathcal{W}(r,n)$ is a subspace of $\mathbb{C}[\mathrm{Gr}(r,n)]$. We will call elements of $\mathcal{W}(r,n)$ *tensor invariants*. An *SL_r-web diagram* is a planar bipartite graph in the disk, with its edges labeled by positive integers so that sum of the labels around each internal vertex is r. We denote by $\mathcal{FS}(\mathrm{SL}_r)$ the vector space of formal sums of SL_r web diagrams. Each web diagram determines an element[1] of $\mathcal{W}(r,n)$, and the resulting map $\mathcal{FS}(\mathrm{SL}_r) \to \mathcal{W}(r,n)$ is surjective.

Webs can be used to study more general tensor invariant spaces, and other homogeneous pieces of $\mathbb{C}[\widetilde{\mathrm{Gr}}(k,n)]$, but in this extended abstract, we focus on a particular situation where our results are easiest to state. Thus, we restrict attention to networks N whose number of boundary vertices n is a multiple of its excedance k, i.e. $n = kr$. For these N, we construct an r-fold boundary measurement map

$$N \mapsto \mathrm{Web}_r(N) \in \mathcal{W}(r,n), \tag{0.2}$$

sending a network to a particular formal sum of webs. Its key feature (our Theorem 1) is that it factors through the boundary measurement map (0.1). That is – if N and N' are two networks with the same boundary measurements, then $\mathrm{Web}_r(N) = \mathrm{Web}_r(N')$ as tensor invariants, even though these will typically be different formal sums of webs. When r is 2 or 3, this reduces to the construction of Temperley-Lieb and web immanants via the double-dimer and triple dimer models studied by the second author [6]. We also show (Theorem 2) that (0.2) induces a natural isomorphism $\mathcal{W}(k,n) \cong \mathcal{W}(r,n)^*$. In fact, it gives rise to an isomorphism $\mathcal{W}(r,n)^* \to \mathcal{W}(k,n) \times \epsilon$ of S_n-modules, where ϵ

[1] In fact, the diagrams defined in this introduction only determine a tensor invariant up to a sign. One can choose signs by picking a *perfect orientation* of this diagram.

is the sign representation. Thus, we get an S_n-equivariant pairing between SL_k and SL_r invariant spaces (unique up to scalars, due to the irreducibility of these S_n-modules).

Now we outline the rest of the abstract. Section 1.1 reviews the boundary measurement map for networks via dimer configurations [6], local moves on networks, and Postnikov's connectedness theorem for networks with the same boundary measurements. Section 1.2 gives the basics of tensor invariants and web diagrams through an example. In Section 2.1 we make our main definition, i.e. the r-fold boundary measurement map (0.2), and introduce the closely related *immanant map*. Section 2.2 discusses one of our main applications. A guiding problem in the history of web combinatorics was to find a complete set of diagrammatic moves describing the kernel of the map $\mathcal{FS}(\mathrm{SL}_r) \to \mathcal{W}(r,n)$. This problem was solved when $r = 3$ by Kuperberg [5], further studied when $r > 3$ in [2, 8], and settled by Cautis-Kamnitzer-Morrison [1]. Our Theorem 3 says that the completeness of the relations [1] follows from our Theorem 1 and Postnikov's connectedness theorem. Finally, in Section 2.3, we make a connection between webs and the celebrated *positroid subvarieties* $\Pi \subset \mathrm{Gr}(k,n)$. For each Π, we explain how our duality identifies (the multilinear part of) $\mathbb{C}[\Pi]$ with a naturally defined subspace of $\mathcal{W}(r,n)$. We believe that this duality hides many surprising relationships between positroids and webs.

1 Background

1.1 Networks, boundary measurements, and Postnikov's Theorem

Let $\mathrm{Gr}(k,n)$ the Grassmannian of k-dimensional subspaces in a fixed n-dimensional complex vector space, and let $\widetilde{\mathrm{Gr}}(k,n)$ denote the affine cone over $\mathrm{Gr}(k,n)$ with respect to the *Plücker embedding* $\mathrm{Gr}(k,n) \subset \mathbb{P}^{\binom{n}{k}-1}$. The affine cone is the subset of $\mathbb{C}^{\binom{n}{k}}$ whose coordinates satisfy the *Plücker relations*. The coordinate ring $\mathbb{C}[\widetilde{\mathrm{Gr}}(k,n)]$ is generated by *Plücker coordinates* $(\Delta_I)_{I \in \binom{[n]}{k}}$; such a coordinate Δ_I can be thought of as a maximal minor of a $k \times n$ matrix using the columns indicated by I.

By a *planar bipartite graph in the disk* we mean a graph G embedded in a closed disk, with its vertices colored in two colors (black and white) such that edges join vertices of opposite color. We suppose that there are exactly n vertices of G on the boundary of the disk, and label these $1,\ldots,n$ in clockwise order. Furthermore, we make the *simplifying assumption* that all of the boundary vertices are black and that each boundary vertex is incident to at most one edge.

By a network N we will mean a planar bipartite graph whose edges have been weighted by nonzero complex numbers. A *dimer configuration* on N (or *almost perfect matching* of N), is a subset π of edges of N that uses each interior vertex exactly once (and uses each boundary vertex one or zero times). The *weight* $\mathrm{wt}(\pi)$ is the product of the weights of the edges used in π. The *boundary subset* $I(\pi) \subset [n]$ is the set of boundary vertices that are used in π. The cardinality $k = |I(\pi)|$ depends only on N (not on the choice of π). Explicitly: $k =$ no. of interior white vertices in N minus no. of interior black vertices. We call the number k the *excedance* of N.

The *boundary measurement* $\Delta_I(N)$ is a weight generating function for dimer configurations with boundary I:

$$\Delta_I(N) = \sum_{\pi:\ I(\pi)=I} \mathrm{wt}(\pi). \tag{1.1}$$

Proposition (Kuo [4], Postnikov-Speyer-Williams [10], Lam [7])**.** *The boundary measurements $(\Delta_I(N)) \in \mathbb{C}^{\binom{n}{k}}$ determine a point $\tilde{X}(N)$ in the affine cone $\widetilde{\mathrm{Gr}}(k,n)$, and thus a point $X(N) \in \mathrm{Gr}(k,n)$ (provided at least one of the Plücker coordinates of $\tilde{X}(N)$ is not zero).*

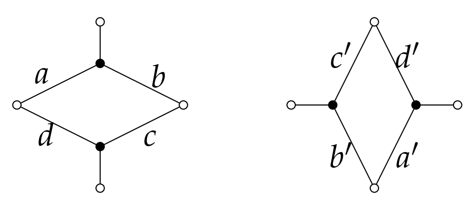
That is, the boundary measurements satisfy the Plücker relations.

It is easy to verify that the following local moves can be applied to a network N to yield a new network N' satisfying $X(N') = X(N)$. Thus the boundary measurements differ by a common scalar α, and we write $\tilde{X}(N) = \alpha\tilde{X}(N')$.

(G) Gauge equivalence: If $e_1, e_2, \ldots, e_d$ are incident to an interior vertex v, we can multiply all of their edge weights by the same constant $\alpha \in \mathbb{C}^*$. The resulting network N' satisfies $\tilde{X}(N) = \alpha\tilde{X}(N')$.

(M1) Spider move, square move, or urban renewal: assuming the leaf edges of the spider have been gauge fixed to 1, the transformation is

$$a' = \frac{a}{ac+bd} \qquad b' = \frac{b}{ac+bd} \qquad c' = \frac{c}{ac+bd} \qquad d' = \frac{d}{ac+bd} \tag{1.2}$$



and $\tilde{X}(N) = (ac+bd)\tilde{X}(N')$.

(M2) Valent two vertex removal. If v has degree two, we can gauge fix both incident edges (v,u) and (v,u') to have weight 1, then contract both edges.

(R1) Multiple edges with same endpoints is the same as one edge with sum of weights.

(R2) Leaf removal: Suppose v is a leaf, and (v,u) the unique edge incident to it. Then we can remove both v and u, and all edges incident to u.

(R3) Dipoles (two degree one vertices joined by an edge) can be removed.

On the other hand, the following is one of the deepest results on the combinatorics of networks:

Theorem (Postnikov [9]). *If N and N' satisfy $\tilde{X}(N) = \tilde{X}(N')$, then they are connected to each other by a finite sequence of these moves. If furthermore both of these networks have the minimal number of faces in their move-equivalence class, then they are connected by the moves (M1) and (M2).*

1.2 Tensor invariants and webs

Let U be an r-dimensional vector space. A *tensor invariant* is an element of the space $\mathcal{W}(r,n) = \mathrm{Hom}_{\mathrm{SL}(U)}(U^{\otimes n}, \mathbb{C})$, i.e. an $\mathrm{SL}(U)$-invariant multilinear function of n vectors $v_1, \ldots, v_n \in U$. Such invariants only exist when $n = kr$ is a multiple of r. In our examples, we assume that we have chosen a basis $E_1 \ldots, E_r$ for U, satisfying $E_1 \wedge \cdots \wedge E_r = 1$.

Webs are convenient way of encoding tensor invariants by planar diagrams. Before making the definition, recall that a *perfect orientation* $\mathcal{O}$ of a planar bipartite graph G is a way of directing the edges in G so that every white vertex has outdegree 1 and every black vertex has indegree 1. We note that if G has excedance k, then $\mathcal{O}$ will have exactly k boundary vertices that are sinks and $n-k$ boundary vertices that are sources.

Definition (SL_r web diagram). Let D be a disk with $n = kr$ boundary vertices. An SL_r web $\hat{W}$ is a perfectly oriented planar bipartite graph in D, with each of its edges e labeled by an integer $m(e) \in [r-1]$, so that the sum of the incoming labels equals the sum of the outgoing labels at every interior vertex. We further require that every boundary source edge has label equal to 1.

To avoid technicalities, we will assume that $\hat{W}$ has been perfectly oriented so that it is without oriented cycles, because it simplifies the association of a tensor invariant to $\hat{W}$. If no such perfect orientation is available, one has to remove oriented cycles by *tagging* certain edges [1].

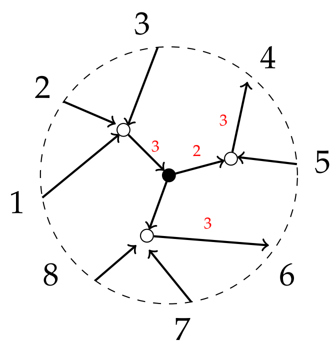
The white and black vertices in a web diagram $\hat{W}$ correspond to the basic $\mathrm{SL}(U)$-invariant tensors

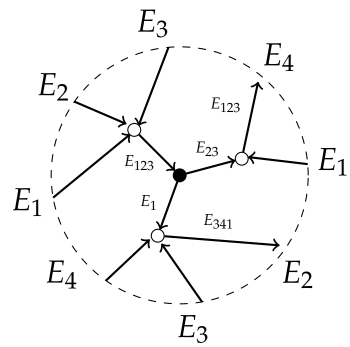
$$\bigwedge^{a_1}(U) \otimes \cdots \otimes \bigwedge^{a_s}(U) \overset{\text{white}}{\to} \bigwedge^{a_1+\cdots+a_s}(U) \quad \text{and} \quad \bigwedge^{a_1+\cdots+a_s}(U) \overset{\text{black}}{\to} \bigwedge^{a_1}(U) \otimes \cdots \otimes \bigwedge^{a_s}(U). \tag{1.3}$$

The first map in (1.3) is the exterior product map, and the second map is the $\mathrm{SL}(U)$-equivariant map that splits up a skew-symmetric tensor $x_1 \wedge \cdots \wedge x_b$ as a signed sum of its constituent parts, where $b = a_1 + \cdots + a_s$.

An edge with label $m(e) = a_i$ in $\hat{W}$ stands for the exterior power $\bigwedge^{a_i}(U)$ in (1.3). To evaluate $\hat{W}$ on some input vectors $(v_1, \ldots, v_n)$, we place the input vector v_i at boundary vertex i and let the vectors flow along the edges of $\hat{W}$, applying the maps (1.3) as indicated by the labels on $\hat{W}$. An input vector v_i sitting at a boundary sink does not go anywhere during this flow. The flow ends with a tensor t_i sitting at the edge for this boundary sink, and we get a number from this edge by taking the pairing $t_i \wedge v_i \in \bigwedge^r(U) \cong \mathbb{C}$. Rather than making these definitions more carefully, we refer the reader to [1], and offer an illustrative example.

Example 1. Consider the following SL_4 web $\hat{W}$ (edges labels equal to 1 are omitted). The underlying graph for $\hat{W}$ has excedance 2, and there are exactly 2 boundary sinks at vertices 4 and 6. Let $E_1, \ldots, E_4$ be standard basis vectors for $U \cong \mathbb{C}^4$. In the second figure, we indicate how to evaluate $\hat{W}$ on the simple tensor $E = E_1 \otimes E_2 \otimes E_3 \otimes E_4 \otimes E_1 \otimes E_2 \otimes E_3 \otimes E_4 \in U^{\otimes 8}$. Begin by placing the ith tensor factor of E at boundary vertex i.





Vectors flow along directed arrows until they hit a black vertex, which in this example happens when the wedge product $E_1 \wedge E_2 \wedge E_3$ is at the black interior vertex of $\hat{W}$. The second map in (1.3) splits this tensor up as a signed sum

$$(E_1 \wedge E_2) \otimes E_3 - (E_1 \wedge E_3) \otimes E_2 + (E_2 \wedge E_3) \otimes E_1, \tag{1.4}$$

but only one of these three terms flows on to pair nontrivially with the vectors at vertices 4 and 6. The flow stops with E_{123} at vertex 4 and E_{341} at vertex 6. We obtain a number at both of these vertices by pairing with corresponding vector from E. That is, at vertex 4 we get a sign of $+1$ from the wedge $(E_1 \wedge E_2 \wedge E_3) \wedge E_4$, and vertex 6 we get $(E_3 \wedge E_4 \wedge E_1) \wedge E_2 = +1$. Therefore, the final value for $\hat{W}|_E$ is $1 \cdot 1 \cdot 1 = 1$, obtained as the product of the signs from boundary vertices 4 and 6 and the interior black vertex.

We remark that, unlike most authors [1, 5], we do not require that the internal vertices of a web diagram are always trivalent. Requiring trivalence is reasonable, because it can be shown that all possible $\mathrm{SL}(U)$-equivariant maps amongst tensor products of $\bigwedge^i(V)$ come from compositions of maps involving three tensor factors, but it seems inconvenient from the networks perspective that we embrace in this paper.

Example 2 (Webs in small rank)**.** Let $v_1, \ldots, v_n \in U$ and $\hat{W}$ be an SL_r web. Think of the vector v_i sitting at boundary vertex i of $\hat{W}$. When $r = 1$, then $U \cong \mathbb{C}$, and $\hat{W}$ can be thought of as a monomial in the vectors $v_1, \ldots, v_n$. When $r = 2$, an SL_2 web $\hat{W}$ consists of a disjoint union of a) oriented cycles of even length, and b) directed paths $v_i \to v_j$ between boundary vertices. Each oriented cycle contributes a multiplicative factor of 2 when $\hat{W}$ is evaluated on $v_1, \ldots, v_n$. Changing the orientation on a path changes the web by a minus sign. Thus, ignoring these signs, and removing all oriented cycles, SL_2 webs are spanned by crossingless matchings on the boundary vertices. In fact – these crossingless matchings are a basis for $\mathcal{W}(2, 2r)$.

A result in similar spirit is true when $r = 3$. In this case, the sign of a web does not depend on the choice of a perfect orientation. Thus, SL_3 webs are typically drawn without directed edges. There are specific diagrammatic rules for expressing any SL_3 in terms of a basis of *non-elliptic webs* [5], i.e. webs that are without 2-valent vertices (thus all edges have label 1), and without interior faces bounded by 4 or fewer sides.

When $r \geq 4$, the existence of a *web basis* satisfying enough "desirable" properties is unknown.

The relation between webs and Grassmannians is as follows: the space $\mathcal{W}(r, n)$ sits inside $\mathbb{C}[\widetilde{\mathrm{Gr}}(r, n)]$ as a certain graded piece. More specifically, there is a $\mathbb{Z}^n$-grading of $\mathbb{C}[\widetilde{\mathrm{Gr}}(r, n)]$ given by the degree in each column. For example, the product of Plücker coordinates $\Delta_{123}\Delta_{256} \in \mathbb{C}[\widetilde{\mathrm{Gr}}(3, 6)]$ has grading $(1, 2, 1, 0, 1, 1)$. The web space $\mathcal{W}(r, n)$ coincides with the graded piece $\mathbb{C}[\widetilde{\mathrm{Gr}}(r, n)]_{(1,\ldots,1)}$ of functions using each column once.

2 Results

2.1 The main construction

Now we are set up to make the connection between almost perfect matchings and webs. Suppose we are given r almost perfect matchings $\pi_1, \ldots, \pi_r$ of N. By superimposing them we naturally obtain the following:

Definition. An *r-weblike subgraph* $W \subset G$ (alternatively, $W \subset N$) is a subgraph of G with each edge e labeled by $m(e) \in [r - 1]$ in such a way that the sum of the labels around each interior vertex is r. The weight of W is the product $\prod_e \mathrm{wt}(e)^{m(e)}$ of the edge weights raised to the corresponding label. Furthermore, each boundary edge must have label 1.

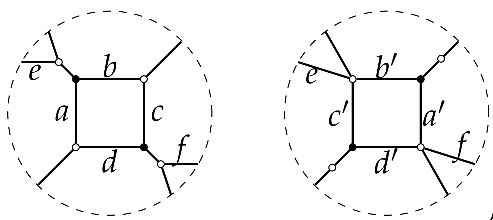
Each r-weblike subgraph determines an SL_r web up to a sign. To fix the signs we can choose a perfect orientation $\mathcal{O}$ on N. If W is an r-weblike subgraph, we get an SL_r web $\hat{W} = \hat{W}(W, \mathcal{O})$ by first directing the edges in $\hat{W}$ as governed by $\mathcal{O}$, and then modifying the edge labels so that if $m(e) = a$ in W and e is directed from white$\to$black in $\mathcal{O}$, then $m(e) = r - a$ in $\hat{W}$. For two different perfect orientations $\mathcal{O}$ and $\mathcal{O}'$, the resulting tensor invariants are equal up to a sign $\hat{W}(W,\mathcal{O}) = \pm\hat{W}(W,\mathcal{O}')$. We show that there is a *canonical choice* of sign for the tensor invariant represented by a weblike subgraph W, and denote the tensor invariant with this sign by boldface $\mathbf{W} \in \mathcal{W}(r,n)$. This canonical choice sign is defined via a certain equation specifying how $\mathbf{W}$ evaluates on tensor products of basis vectors [2].

With this in mind, we can make our main definition:

$$\mathrm{Web}_r(N,\mathcal{O}) = \sum_{W \subset N} \mathrm{wt}(W)\mathbf{W} \in \mathcal{W}(r,n). \tag{2.1}$$

It is a $\mathbb{C}$-linear combination of the boldface versions of the r-weblike subgraphs of N, with the $\mathrm{wt}(W)$'s serving as coefficients. We refer to $\mathrm{Web}_r(N)$ as the *r-fold boundary measurement of N*.

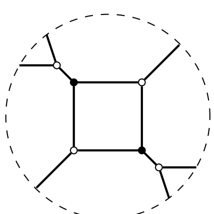
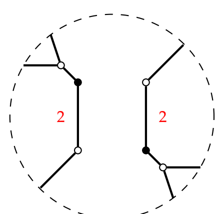
Example 3. Consider the pair of networks N and N' given by

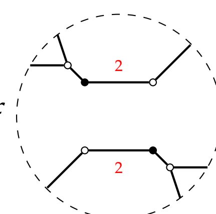


,

where a', b', c', d' are related to a, b, c, d according to the spider move (1.2).

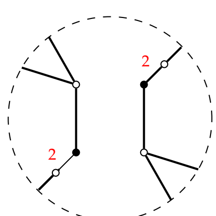
Their parameters are $k = 2$ and $n = 6$, and the letters $a, a', b, b', \ldots, e, f \in \mathbb{C}^\times$ denote edge weights. The network N has three r-weblike subgraphs, and the 3-fold boundary measurement $\mathrm{Web}_3(N)$ is a linear combination

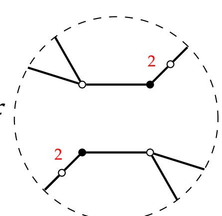
$abcdef$  $+\, a^2c^2ef$ 

$+\, b^2d^2ef$ 

(2.2)

with coefficients depending on $a, \ldots, f$. On the other hand, $\mathrm{Web}_r(N')$ is a linear combination of two webs

$a'c'ef$ 

$+\, b'd'ef$ 

. (2.3)

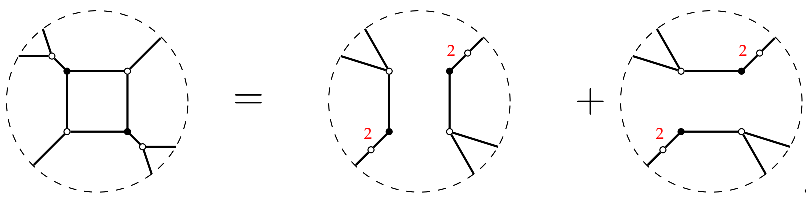
[2]A pesky detail is that it can happen that $\mathbf{W} = -\hat{W}(W,\mathcal{O})$ for every perfect orientation $\mathcal{O}$, i.e. the correct choice of sign is not given by *any* perfect orientation.

We can now state our main theorem on the r-fold boundary measurements (2.1):

Theorem 1. *If N and N' are two networks satisfying $\tilde{X}(N) = \tilde{X}(N')$, then* $\mathrm{Web}_r(N) = \mathrm{Web}_r(N') \in \mathcal{W}(r,n)$.

That is, the r-fold boundary measurement map factors through the boundary measurement map.

Example 4. Continuing with Example 3, recall that a', b', c', d' are related to a, b, c, d by (1.2). The networks N and N' are related by (M1), and $\tilde{X}(N) = (ac+bd)\tilde{X}(N')$. It follows that $\mathrm{Web}_3(N;\lambda) = (ac+bd)^3\, \mathrm{Web}_3(N';\lambda)$. By equating the coefficient of $abcdef$ in (2.2) with the coefficient of $\frac{abcdef}{(ac+bd)^3}$ in (2.3), we deduce the *square move* for SL_3 webs:



(2.4)

Theorem 1 leads to an interesting relationship between SL_k and SL_r web spaces. Let $\varphi \in \mathcal{W}(r,n)^*$ be a functional on the SL_r web space. Let N be a network with excedance k and $n = kr$ boundary vertices. By Theorem 1, the function $\tilde{X}(N) \mapsto \mathrm{Web}_r(N)$ is well-defined (it doesn't depend on the choice of network N representing $\tilde{X}(N) \in \widetilde{\mathrm{Gr}}(k,n)$). This map is only defined on the part of $\widetilde{\mathrm{Gr}}(k,n)$ swept out by $\tilde{X}(N)$'s, but we prove that it extends to a regular function on $\widetilde{\mathrm{Gr}}(k,n)$. We obtain in this way a linear map $\mathcal{W}(r,n)^* \to \mathcal{W}(k,n) \subset \mathbb{C}[\widetilde{\mathrm{Gr}}(k,n)]$, which we call the *immanant map*.

We prove that the immanant map is an isomorphism. Moreover, both tensor invariant spaces $\mathcal{W}(k,n)$ and $\mathcal{W}(r,n)$ carry an action of the symmetric group by permutations of the vectors. These S_n-modules are irreducible – corresponding to the Specht modules of rectangular shapes $k \times r$ and $r \times k$ respectively – and are related to each other by tensoring with the sign representation ϵ.

Theorem 2. *The immanant map $\mathcal{W}(r,n)^* \to \mathcal{W}(k,n) \otimes \epsilon$ is an isomorphism of S_n-modules.*

We note that Theorem 2 does not seem immediately obvious, since the symmetric group does not act in a natural way on networks.

When r is 2 or 3, the second author previously defined SL_2 and SL_3 *web immanants* [6]. These web immanants are obtained by applying the immanant map to the functional $\varphi_W \in \mathcal{W}(r,n)^*$, where W is a basis web and φ_W is the dual basis element. In the case that $r \geq 4$, since we are without a notion of a web basis, we believe that $\mathrm{Web}_r(N)$ is the more fundamental object.

2.2 Deducing skein relations from moves on networks

According to Theorem 1, the element $\mathrm{Web}_r(N)$ only depends on $\tilde{X}(N)$. Thus it is unchanged when we perform the local moves from Section 1.1 to N. However, the *expression* for $\mathrm{Web}_r(N)$ inside $\mathcal{FS}$ typically changes after each local move. In this way, we obtain relations amongst SL_r webs.

As it turns out, the diagrammatic relations on webs we obtain in this way are exactly the Cautis-Kamnitzer-Morrison relations [1]. For example, by souping up Example 2, one sees that the spider move (M1) for networks encodes the square move for SL_r webs [1, Equation 2.10]. In this way, one could obtain a different proof of Theorem 1.

In our paper, we employ a different perspective: we give what we consider to be the "right" proof of Theorem 1. Then, we give an "abstract" argument, based on our duality results, that the diagrammatic relations amongst webs encoded by the local moves on networks are a complete set of relations amongst webs.

Theorem 3. *The relations amongst networks imposed by Theorem 1 generate the kernel* $\mathcal{FS}(SL_r) \to \mathcal{W}(r,n)$.

This is an independent proof of the sufficiency of the Cautis-Kamnitzer-Morrison relations for SL_r webs. We note that [1] works in the generality of representations of the quantum group $U_q(\mathfrak{sl}_n)$, whereas our proof currently only makes sense when $q = 1$.

2.3 Positroids and webs

Our networks N are closely related to a special stratification of the Grassmannian by *positroid varieties*. The survey [7] outlines the many senses in which positroids (and positroid varieties) are better behaved than matroids (and matroid varieties).

For any point $x \in \mathrm{Gr}(k,n)$, the *matroid* of x is the realizable matroid $\mathcal{M}(x)$ formed by the subsets $I \in \binom{[n]}{k}$ such that $\Delta_I(x) \neq 0$. The *matroid variety* associated to a matroid $\mathcal{M}$ is the closure in $\mathrm{Gr}(k,n)$ of the set of points whose matroid is $\mathcal{M}$.

The *totally nonnegative Grassmannian* $\mathrm{Gr}(k,n)_{\geq 0} \subset \mathrm{Gr}(k,n;\mathbb{R})$ consists of points that can be given by $k \times n$ matrices with real entries, whose Plücker coordinates are all ≥ 0. A matroid is a *positroid* if it is the matroid of a totally nonnegative point $x \in \mathrm{Gr}(k,n)_{\geq 0}$. The corresponding matroid variety $\Pi = \Pi_{\mathcal{M}}$ is called a *positroid variety*.

Theorem (Postnikov, Knutson-Lam-Speyer [3]). *Let G be a planar bipartite graph in the disk. As N varies over networks with graph G and edge weights in $\mathbb{C}^*$, the boundary measurements $X(N)$ sweep out a Zariski dense subset of a single positroid variety $\Pi = \Pi(G)$. Furthermore, every positroid variety Π arises in this way from some planar bipartite graph G.*

Let $\mathcal{M}$ be a positroid. Let $\mathcal{I}(\Pi) = \{\Delta_I : I \notin M\}$. Then $\mathcal{I}(\Pi)$ is the homogeneous prime ideal of $\Pi_{\mathcal{M}}$ [3], i.e. the surjection $\mathbb{C}[\widetilde{\mathrm{Gr}}(k,n)] \twoheadrightarrow \mathbb{C}[\Pi]$ has kernel $\mathcal{I}(\Pi)$. The grading on $\mathbb{C}[\widetilde{\mathrm{Gr}}(k,n)]$ descends to a grading on $\mathbb{C}[\Pi]$.

Since $\mathcal{W}(k,n)$ and $\mathcal{W}(r,n)$ are dual, the map $\mathbb{C}[\widetilde{\mathrm{Gr}}(k,n)]_{(1,\ldots,1)} \twoheadrightarrow \mathbb{C}[\Pi]_{(1,\ldots,1)}$ should be dual to an inclusion of some subspace into $\mathcal{W}(r,n)$. Let us identify this subspace:

Let $x \in \mathcal{W}(r,n)$, thought of as an SL_r-invariant function of vectors $v_1,\ldots,v_n \in U$. For any $I \in \binom{[n]}{k}$, there is a *partial evaluation map* $\mathcal{W}(r,n) \to \mathcal{W}(r-1,n-k)$, denoted $x \mapsto x\big|_I$, obtained by specializing $v_i = E_r$ for each $i \in I$. The resulting function is an SL_{r-1}-invariant function of $n-k$ vectors. We denote by $\mathcal{W}(r,n)_{\mathcal{M}} \subset \mathcal{W}(r,n)$ the subspace of $\mathcal{W}(r,n)$ consisting of tensors x whose partial evaluation $x\big|_I$ is 0 *for every* $I \notin \mathcal{M}$.

Theorem 4. *The surjection* $\mathbb{C}[\widetilde{\mathrm{Gr}}(k,n)]_{(1,\ldots,1)} \twoheadrightarrow \mathbb{C}[\Pi]_{(1,\ldots,1)}$ *is dual to the inclusion* $\mathcal{W}(r,n)_{\mathcal{M}} \hookrightarrow \mathcal{W}(r,n)$*. In particular* $\dim(\mathbb{C}[\Pi]_{(1,\ldots,1)})$ *equals* $\dim(\mathcal{W}(r,n)(\mathcal{M}))$*. If G is a planar bipartite graph with positroid $\mathcal{M}$, then subspace $\mathcal{W}(r,n)_{\mathcal{M}}$ is spanned by either of the following sets: i) the elements* $\mathrm{Web}_r(N)$*, as N varies over networks whose underlying graph is G, or ii) the r-weblike subgraphs of G.*

This characterization of $\dim(\mathbb{C}[\Pi]_{(1,\ldots,1)})$ may be easier to work with than the characterization using promotion and cyclic Demazure crystals given in [7, Section 12].

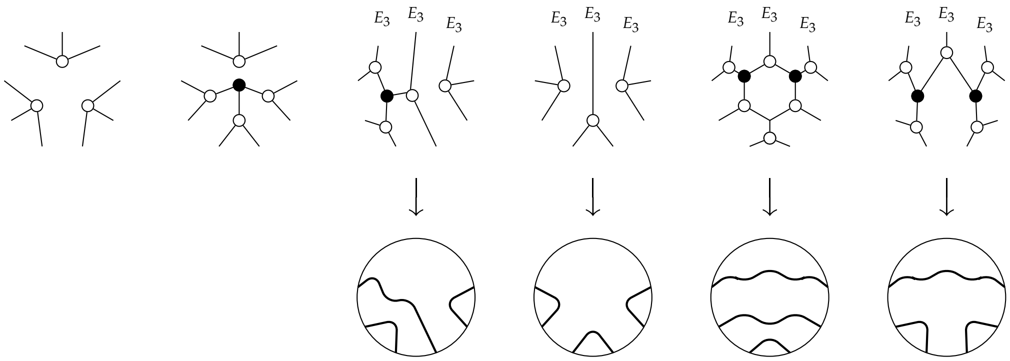
Example 5. Focus on the space $\mathcal{W}(3,9)$ of SL_3-invariant multilinear functions of $v_1,\ldots,v_9$. We have $\dim(\mathcal{W}(3,9)) = \#$ of 3×3 SYT's $= 42$. The basis webs (up to rotation) are listed in (2.5). Let $\mathcal{M}$ be the positroid $\mathcal{M} = \binom{[9]}{3} \setminus \{\Delta_{789}\}$. The subspace $\mathcal{W}(3,9)_{\mathcal{M}}$ is the kernel of the map $\mathcal{W}(3,9) \to \mathcal{W}(2,6)$ specializing $v_7 = v_8 = v_9 = E_3$. A web W is in $\mathcal{W}(3,9)_{\mathcal{M}}$ exactly when W has a fork connecting vertices 7 and 8 or 8 and 9. By examining the 42 webs (2.5), we see there are exactly 37 webs that are in $\mathcal{W}(3,9)_{\mathcal{M}}$, and five webs that are not. From Theorem 4, $\dim(\mathbb{C}[\Pi]_{(1,\ldots,1)}) = 37$.

In (2.5), we placed E_3's at the three consecutive boundary vertices without forks. This gives 4 of the webs W such that $W|_{\{7,8,9\}} \neq 0$; the 5th web is obtained by reflecting the third web in (2.5) along the vertical axis. Once the boundary E_3's are placed, and certain interior E_3's are forced, the leftover interior edges must be labeled by E_1's and E_2's. This produces an SL_2 web, as drawn in (2.5). The 5 resulting crossingless matchings on six vertices are the 5 basis webs for $\mathcal{W}(2,6)$. Thus, no nontrivial linear combination of these five SL_3 webs produces an element of $\mathcal{W}(3,9)_{\mathcal{M}}$, i.e. $\mathcal{W}(3,9)_{\mathcal{M}}$ is the span of the 37 webs that vanish under partial evaluation.

Acknowledgements

This work began during the Snowbird '14 cluster algebras MRC. We thank our fellow group members (Darlayne Addabbo, Eric Bucher, Sam Clearman, Laura Escobar, Suho

Oh, Hannah Vogel) for their help on this project.



(2.5)

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